University of Nottingham
Department of Mechanical Engineering

MM2MS2 Mechanics of Solids 2
Exercise Sheet 5 - Elastic-Plastic Deformations Solutions

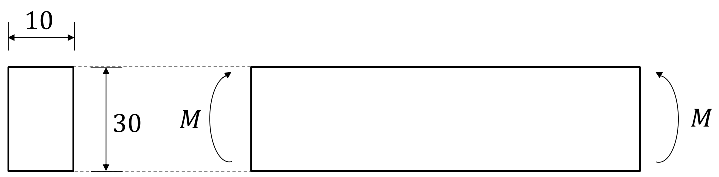
For all questions assume E = 200GPa; G = 75GPa; σ_y = 250MPa for steel.

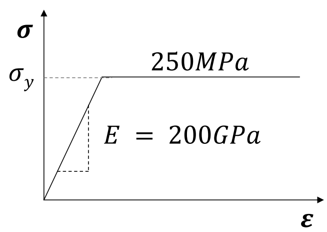
1. A steel beam of rectangular section 10mm x 30mm is subjected to pure bending in a plane parallel to the 30mm faces. Ideal elastic-plastic behaviour may be assumed. Calculate the bending moment, M, necessary for:

   (a) the onset of yield,
   (b) complete yield through the section.

[Ans: a) 375Nm, b) 562.5Nm]

Solution 1





$$I = \frac{bd^3}{12} = \frac{10 \times 30^3}{12} = 22500mm^4$$

(a) Onset of yield:

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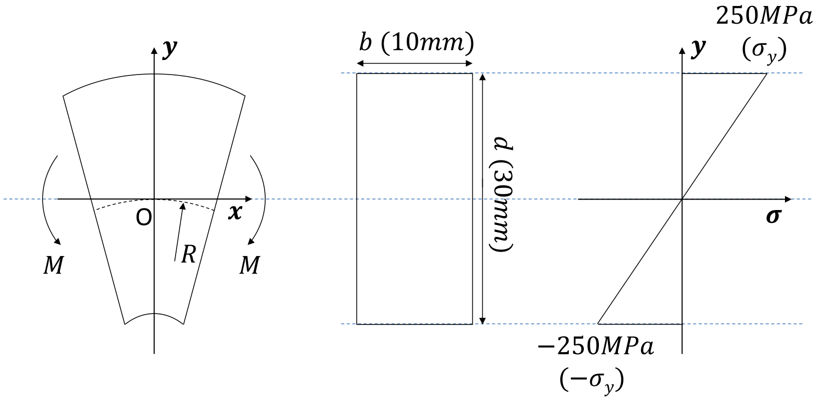

Variation of stress with y:

- For $-15 < y < 15$, $\sigma = \frac{250}{15} yMPa$

Behaviour is all elastic and therefore:

$$\frac{M_y}{I} = \frac{\sigma_y}{y}$$

(where M_y is the moment required to cause yielding)

$$\therefore M_y = \frac{\sigma_y \times I}{y} = \frac{\sigma_y \times I}{d/_2} = \frac{250MPa \times 22500mm^4}{\frac{30}{2}mm} = 375000Nmm = \mathbf{375Nm}$$

(b) Complete yield through the section:

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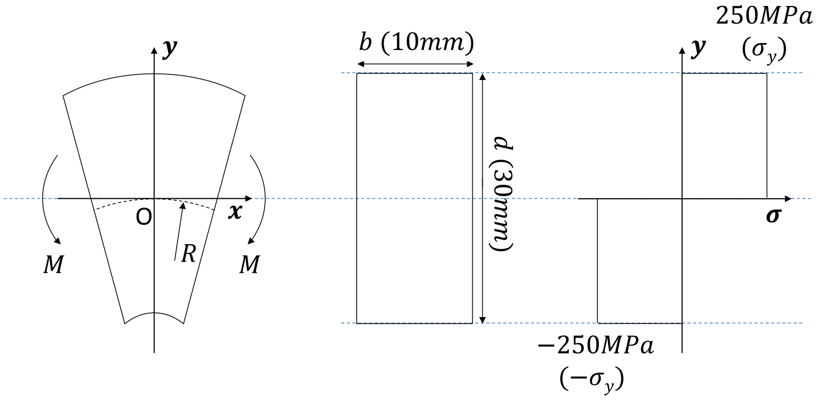

Section of the beam in bending

Cross section of the beam

Stress distribution through the beam cross section

Variation of stress with y:

- For $0 < y < 15$, $\sigma = 250MPa$
- For $-15 < y < 0$, $\sigma = -250MPa$

Moment equilibrium

$$M = \int_A y\sigma dA$$

Since the area under the graph can be calculated simply, the right hand side of the above can be rewritten as:

$$M = 2\left(\frac{d}{4} \times \sigma_y \times b\frac{d}{2}\right)$$

$$\therefore M = 2 \times \left(\frac{30}{4} \times 250 \times 10\frac{30}{2}\right) = 562500Nmm = \mathbf{562.5Nm}$$

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2. A bending moment of 400Nm is applied to the beam of question 1. (a) Calculate the surface strain. (b) Determine the maximum residual stress when the bending moment is removed.

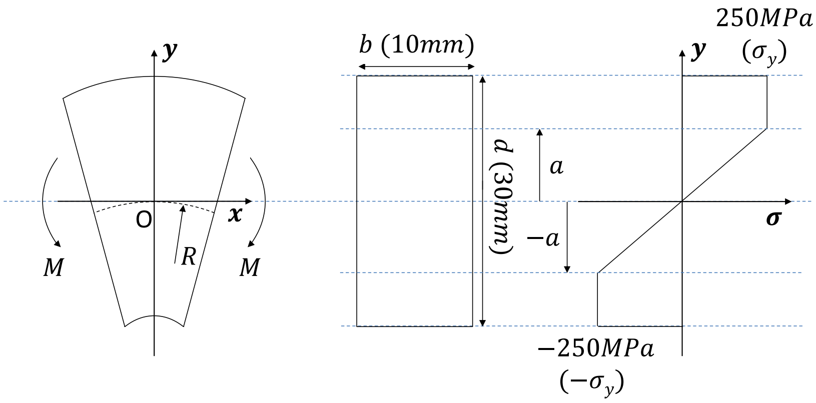
[Ans: a) 0.134 %, b) ±16.7 MPa]

Solution 2

(a)

$M = 400Nm$

$M > M_y$, therefore yielding occurs.



Section of the beam in bending

Cross section of the beam

Stress distribution through the beam cross section

Variation of stress with y:

- For a $< y < 15$, $\sigma = 250MPa$
- For $-$a $< y < a$, $\sigma = \frac{250}{a} yMPa$
- For $-15 < y < -a$, $\sigma = -250MPa$

Moment equilibrium

$$M = \int_A y\sigma dA = \int_{-d/2}^{d/2} y\sigma b dy = 2\int_0^{d/2} y\sigma b dy$$

$$\therefore M = 2\left\{\int_0^a y\left(\frac{250}{a}y\right) bdy + \int_a^{d/2} y(250)bdy\right\} = 2 \times 250b\left\{\int_0^a \frac{y^2}{a}dy + \int_a^{d/2} ydy\right\}$$

$$= 500b\left\{\left[\frac{y^3}{3a}\right]_0^a + \left[\frac{y^2}{2}\right]_a^{d/2}\right\} = 500b\left\{\frac{a^2}{3} + \frac{d^2}{8} - \frac{a^2}{2}\right\}$$

$$\therefore M = 500b\left\{\frac{d^2}{8} - \frac{a^2}{6}\right\}$$

$$\therefore 400000Nmm = 500b\left\{\frac{d^2}{8} - \frac{a^2}{6}\right\} = 500 \times 10\left\{\frac{30^2}{8} - \frac{a^2}{6}\right\}$$

$$\therefore a = 13.96mm$$

Compatibility

Beam bending equation:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \frac{y}{R} = \varepsilon\left(= \frac{\sigma}{E}\right)$$

$$\therefore R = \frac{y}{\varepsilon} \qquad \text{(i)}$$

σ-ε relationship

At $y = a$ (outermost elastic point), $\sigma = \sigma_y$ and $\varepsilon = \varepsilon_y$ and the elastic relation, $\sigma_y = E\varepsilon_y$. Therefore at $y = 13.96\ (= a)$,

$$\varepsilon_y = \frac{\sigma_y}{E} = \frac{250}{200 \times 10^3} = 1.25 \times 10^{-3}$$

Applying this to (i) gives:

$$R = \frac{y\ (at\ a)}{\varepsilon} = \frac{13.96}{1.25 \times 10^{-3}}$$

$$\therefore R = 11168mm = 11.168m$$

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Surface strain

$$Surface\ strain = \frac{y}{R} = \frac{d}{2R} = \frac{30}{2 \times 11168} = 0.00134 = \mathbf{0.134}\%$$

(b)

Unloading (assuming all elastic)

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta \sigma}{y}$$

Max change in σ will occur at $y = y_{max}$ $(= \pm 15mm)$.

$$\therefore \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-400000 \times \pm 15}{22500} = \mp 266.67MPa$$

i.e.:

$$at\ y = 15mm, \therefore \Delta\sigma_{max}^{el} = -266.67MPa$$

$$and\ at\ y = -15mm, \therefore \Delta\sigma_{max}^{el} = 266.67MPa$$

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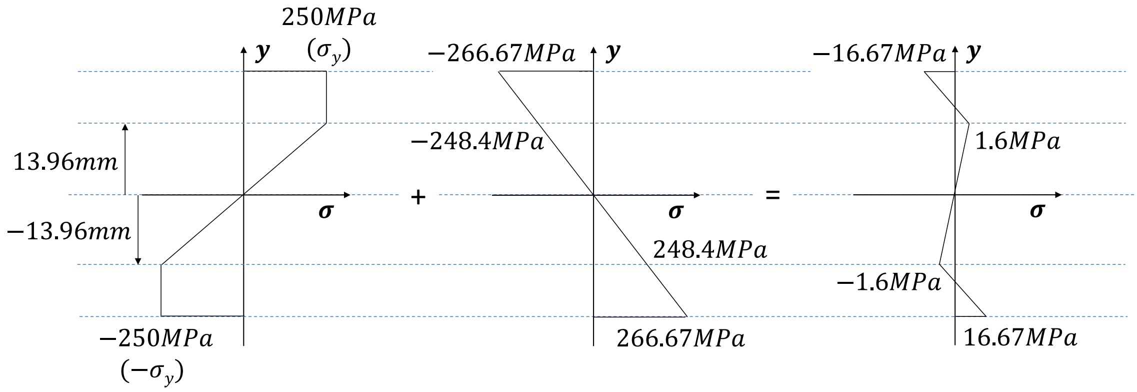

Loading stress distribution

(Elastic) unloading stress distribution

Residual stress distribution

Interpolation of (elastic) unloading line:

At $y = 15mm, \sigma = -266.67MPa$

$$y = m\sigma + c$$

$$\therefore 15 = \mathrm{m} \times -266.67 + 0$$

$$\therefore \mathrm{m} = 0.0562$$

At $y = 13.96mm$, $13.96 = 0.0562 \times \sigma$

$$\therefore \sigma = 248.4MPa$$

Therefore maximum residual stress:

$$\sigma_{res}^{max} = \pm\mathbf{16.67}\boldsymbol{MPa}$$

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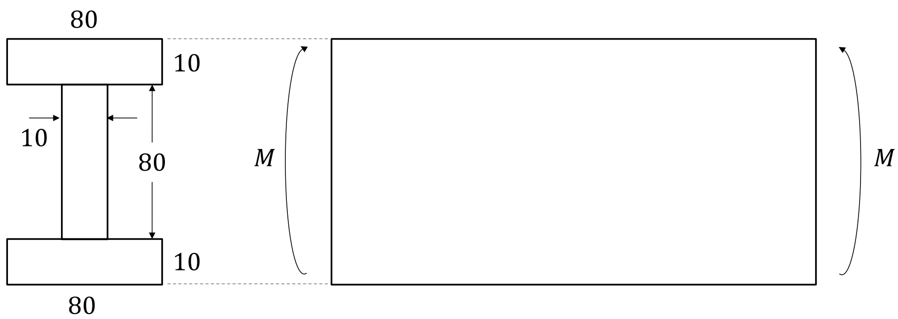
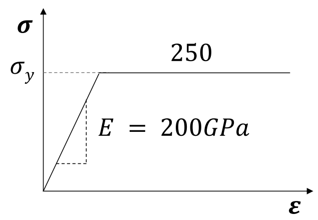
3. The web and flanges of a straight I-section steel beam are 80 mm wide and 10 mm thick. The beam is loaded in pure bending in the plane of the web until the whole of each flange has yielded but the whole of the web remains elastic. Calculate the residual curvature in the unloaded beam. Assume ideal elastic-plastic behaviour.

[Ans: 3.175 x 10^{-3} m^{-1}]

Solution 3





$$I = \frac{b_o d_o^3}{12} - 2 \times \left(\frac{b_i d_i^3}{12}\right) = \frac{80 \times 100^3}{12} - 2 \times \left(\frac{35 \times 80^3}{12}\right) = 3680000.03mm^4$$

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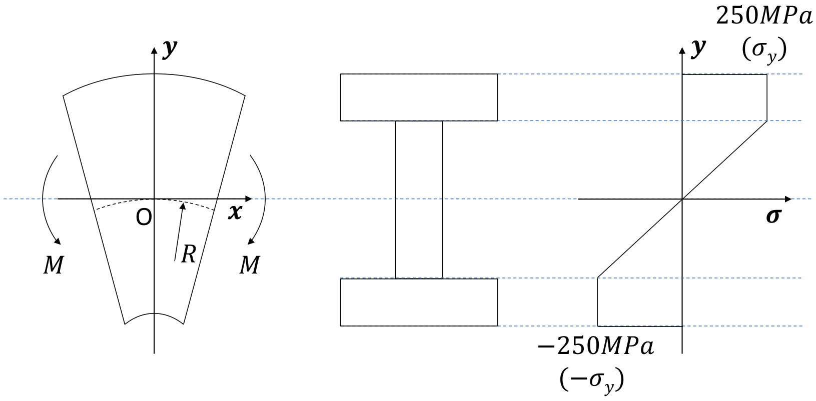
Application of bending load



Section of the beam in bending

Cross section of the beam

Stress distribution through the beam cross section

Variation of stress with y:
- For $40 < y < 50$, $\sigma = 250MPa$
- For $-40 < y < 40$, $\sigma = \frac{250}{40} yMPa$
- For $-50 < y < -40$, $\sigma = -250MPa$

Moment equilibrium

$$M = \int_A y\sigma dA = \int_{-50}^{50} y\sigma b dy = 2\int_0^{50} y\sigma b dy$$

$$\therefore M = 2\left\{\int_0^{40} y \times \left(\frac{250}{40}y\right) \times 10dy + \int_{40}^{50} y \times 250 \times 80dy\right\} = 2 \times 250\left\{\frac{1}{4}\int_0^{40} y^2 dy + 80\int_{40}^{50} ydy\right\}$$

$$= 500\left\{\frac{1}{4}\left[\frac{y^3}{3}\right]_0^{40} + 80\left[\frac{y^2}{2}\right]_{40}^{50}\right\} = 500\left\{\frac{1}{4}\left(\frac{40^3}{3} - \frac{0^3}{3}\right) + 80\left(\frac{50^2}{2} - \frac{40^2}{2}\right)\right\}$$

$$= 500\left\{\frac{1}{4} \times 21333.3 + 80 \times (1250 - 800)\right\} = 500 \times 41333.325$$

$$\therefore M = 20666662.5Nmm = 20.67kNm$$

Compatibility

Beam bending equation:

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$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \frac{y}{R} = \varepsilon \left(= \frac{\sigma}{E}\right)$$

$$\therefore R = \frac{y}{\varepsilon} \qquad \text{(i)}$$

$\sigma - \varepsilon$ relationship

At $y = \pm 40mm$ (outermost elastic point), $\sigma = \pm\sigma_y$ and $\varepsilon = \pm\varepsilon_y$ and the elastic relation:

$$\therefore \pm\varepsilon_y = \frac{\pm\sigma_y}{E} \;(still\ elastic)$$

$$\therefore \pm\varepsilon_y = \frac{\pm 250 \times 10^6}{200 \times 10^9}$$

$$\therefore \varepsilon_y = 1.25 \times 10^{-3} \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$R = \frac{40}{1.25 \times 10^{-3}}$$

$$\therefore R = 32000mm = 32m$$

Unloading (assuming all elastic)

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta\sigma}{y}$$

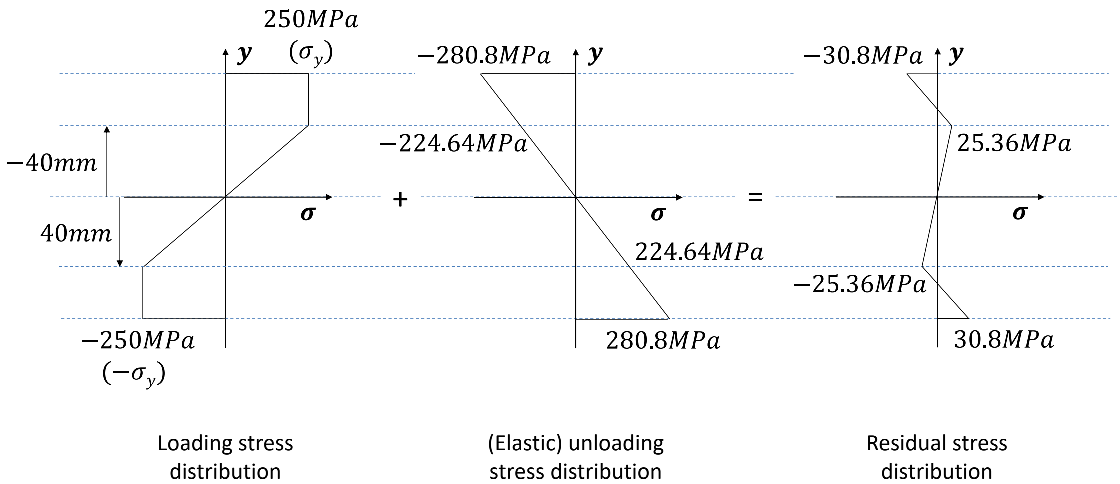
Max change in σ ($\Delta\sigma$) will occur at $y = ymax$ ($= \pm 15mm$).

$$\therefore \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-20666662.5 \times \pm 50}{3680000.03} = \mp 280.8MPa$$

i.e.:

$$at\ y = 50mm, \therefore \Delta\sigma_{max}^{el} = -280.8MPa$$

$$and\ at\ y = -50mm, \therefore \Delta\sigma_{max}^{el} = 280.8MPa$$



Interpolation of (elastic) unloading line:
At $y = 50mm$, $\sigma = -280.8MPa$

$$y = m\sigma + c$$
$$\therefore 50 = \mathrm{m} \times -280.8 + 0$$
$$\therefore \mathrm{m} = 0.178$$

At $y = 40mm$, $40 = 0.178 \times \sigma$

$$\therefore \sigma = 224.64MPa$$

Note that at y = 40mm there has been no plastic deformation.

Therefore, from compatibility:

$$\varepsilon_{residual} = \frac{y}{R_{residual}} \quad \text{(iii)}$$

And from the stress-strain relationship:

$$\varepsilon_{residual} = \frac{\sigma_{residual}}{E} = \frac{25.36 \times 10^6}{200 \times 10^9}$$

$$\therefore \varepsilon_{residual} = 0.000127$$

which in (iii) gives:

$$0.000127 = \frac{40}{R_{residual}}$$

$$\therefore R_{residual} = 314960.63mm = 314.96m$$

Therefore:

$$Residual\ curvature, K_{residual} = \frac{1}{R_{residual}} = \frac{1}{314.96} = \mathbf{3.175 \times 10^{-3} m^{-1}}$$

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4. A 10mm diameter steel rod, formed to the U-shape ABCD, as shown in Fig Q4, is stress free. The leg AB is restrained to remain vertical whilst the leg CD rotates through 90°. This twists BC, which has an effective length, l. When CD is released, calculate:

   (a) the minimum value of l at which CD returns to its original position (angular spring-back of 90°),
   (b) the angular spring-back of CD if $l = 200mm$.

   Assume ideal elastic-plastic material behaviour.

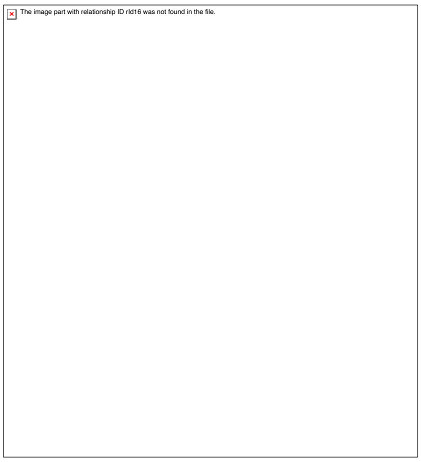

Fig Q4

[Ans: a) Tresca: L = 4.703m, von Mises: L = 4.091m; b) Tresca: 5.093°, von Mises: 5.878°]

Solution 4

$$J = \frac{\pi R^4}{2} = \frac{\pi \times 5^4}{2} = 981.75mm^4$$

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(a) Onset of yield

The shorter the length of section BC, the more likely it is to yield. Therefore, the minimum length of section BC, l_{min}, to avoid yielding of the material will occur at the point at which the onset of yield is about to occur (i.e. at τ_y).

The image part with relationship ID rId16 was not found in the file.

Behaviour is all elastic and therefore (compatibility):

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{l} \qquad \text{(i)}$$

And (stress-strain relationship):

$$\tau = G\gamma \qquad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$l = \frac{R\theta}{\gamma} \qquad \text{(iii)}$$

At the point of yield (i), (ii) and (iii) become:

$$\frac{T_y}{J} = \frac{\tau_y}{R} = \frac{G\theta}{l_{min}} \qquad \text{(iv)}$$

$$\tau_y = G\gamma_y \qquad \text{(v)}$$

$$l_{min} = \frac{R\theta}{\gamma_y} \tag{vi}$$

Therefore from (v):

$$\gamma_y = \frac{\tau_y}{G} \tag{vii}$$

Tresca

$$\tau_y = \frac{\sigma_y}{2} = \frac{250MPa}{2} = 125MPa \tag{viii}$$

Substituting (viii) into (vii):

$$\gamma_y = \frac{125 \times 10^6}{75 \times 10^9} = 0.00167 \tag{ix}$$

Substituting (ix) into (vi) gives:

$$\boxed{l_{min} = \frac{5 \times \frac{\pi}{2}}{0.00167} = 4702.98mm = \mathbf{4.703m}}$$

von Mises

The full form of the von Mises yield criterion is given as:

$$\sigma_v = \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6({\sigma_{12}}^2 + {\sigma_{23}}^2 + {\sigma_{31}}^2)}{2}} \tag{x}$$

where σ_{11} is the direct stress in the x-direction and can be re-written as σ_x, similarly $\sigma_{22} = \sigma_y$ and $\sigma_{33} = \sigma_z$, σ_{12} is the shear stress on the x-y plane and can be re-written as τ_{xy}, and similarly $\sigma_{23} = \sigma_{yz}$ and $\sigma_{31} = \sigma_{zx}$. Therefore (10) can be re-written as:

$$\sigma_v = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6({\tau_{xy}}^2 + {\tau_{yz}}^2 + {\tau_{zx}}^2)}{2}} \tag{xi}$$

In the case of pure shear, $\tau_{xy} = \tau_{yx} \neq 0$, while $\sigma_x = \sigma_y = \sigma_z = \tau_{yz} = \tau_{zy} = \tau_{zx} = \tau_{xz} = 0$). Therefore (x) becomes:

$$\sigma_v = \sqrt{\frac{6{\tau_{xy}}^2}{2}} = \sqrt{3}\tau_{xy}$$

At the point of yield $\sigma_v = \sigma_y$ and therefore xii can be written as:

$$\tau_{xy} = \frac{\sigma_y}{\sqrt{3}}$$

$$\therefore \tau_y = \frac{\sigma_y}{\sqrt{3}} = \frac{250MPa}{\sqrt{3}} = 144.25MPa \qquad \text{(xii)}$$

Therefore from (vii):

$$\gamma_y = \frac{144.25 \times 10^6}{75 \times 10^9} = 0.00192 \qquad \text{(xiii)}$$

Substituting (xiii) into (vi) gives:

$$l_{min} = \frac{5 \times \frac{\pi}{2}}{0.00192} = 4090.62mm = \mathbf{4.091m}$$

(b) $l = 200mm$

$$Angular\ Springback = Applied\ \theta - Residual\ \theta$$

From (iii), at point of yield:

$$r_y = \frac{l\gamma_y}{\theta} = \frac{200 \times 0.00167}{\frac{\pi}{2}} = 0.213mm$$

where $\tau_y = 125MPa$ and $\gamma_y = 0.00167$ for **Tresca** (from (viii) and (ix)). Therefore yield occurs at a radius of 0.213mm as shown in the following figure.

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The image part with relationship ID rId16 was not found in the file.

Torque Equilibrium

$$T = \int_A r\tau dA = \int_0^R r\tau \times 2\pi r dr = 2\pi \int_0^R \tau r^2 dr$$

$$= 2\pi \int_0^{0.213} \frac{125}{0.213} r^3 dr + 2\pi \int_{0.213}^{5} 125 r^2 dr = 250\pi \left(\int_0^{0.213} \frac{r^3}{0.213} dr + \int_{0.213}^{5} r^2 dr \right)$$

$$= 250\pi \left(\left[\frac{r^4}{0.852} \right]_0^{0.213} + \left[\frac{r^3}{3} \right]_{0.213}^{5} \right) = 250\pi \left(\left(\frac{0.213^4}{0.852} - \frac{0^4}{0.852} \right) + \left(\frac{5^3}{3} - \frac{0.213^3}{3} \right) \right)$$

$$\therefore T = 32726.91 Nmm = 32.73 Nm$$

Unloading (assuming all elastic)

From (i):

$$\theta = \frac{Tl}{GJ} = \frac{32726.91 \times 200}{75 \times 10^3 \times 981.75} = 0.0889 rad = \mathbf{5.093°}$$

For **von Mises**, $\tau_y = 144.25 MPa$ and $\gamma_y = 0.00192$ (from (xii) and (xiiii)), therefore:

$$r_y = \frac{l\gamma_y}{\theta} = \frac{200 \times 0.00192}{\frac{\pi}{2}} = 0.244mm$$

Therefore yield occurs at a radius of 0.244mm as shown in the following figure.

Torque Equilibrium

$$T = \int_A r\tau dA = \int_0^R r\tau \times 2\pi r dr = 2\pi \int_0^R \tau r^2 dr$$

$$= 2\pi \int_0^{0.244} \frac{144.25}{0.244} r^3 dr + 2\pi \int_{0.244}^{5} 144.25 r^2 dr = 288.5\pi \left(\int_0^{0.244} \frac{r^3}{0.244} dr + \int_{0.244}^{5} r^2 dr \right)$$

$$= 288.5\pi \left(\left[\frac{r^4}{0.976} \right]_0^{0.244} + \left[\frac{r^3}{3} \right]_{0.244}^{5} \right) = 288.5\pi \left(\left(\frac{0.244^4}{0.976} - \frac{0^4}{0.976} \right) + \left(\frac{5^3}{3} - \frac{0.244^3}{3} \right) \right)$$

$$\therefore T = 37766.49Nmm = 37.76Nm$$

Unloading (assuming all elastic)

From (i):

$$\theta = \frac{Tl}{GJ} = \frac{37766.49 \times 200}{75 \times 10^3 \times 981.75} = 0.1026rad = \mathbf{5.878°}$$